Resolving New Keynesian Anomalies with Wealth in the Utility Function: Online Appendices

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Appendix A. Formal derivation of Euler equation & Phillips curve

We derive the two differential equations that describe the equilibrium of the New Keynesian model with wealth in the utility function: the Phillips curve, given by (1); and the Euler equation, given by (4).

A.1. Household's problem

We begin by solving household j's problem. The current-value Hamiltonian of the problem is

$$\begin{aligned}\mathcal{H}_j &= \frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\,dk\right) + u\left(\frac{b_j(t)-b(t)}{p(t)}\right) - \frac{\kappa}{a}y_j^d(p_j(t),t) - \frac{\gamma}{2}\pi_j(t)^2\\ &+ \mathcal{A}_j(t)\left[i^h(t)b_j(t) + p_j(t)y_j^d(p_j(t),t) - \int_0^1 p_k(t)c_{jk}(t)\,dk - \tau(t)\right] + \mathcal{B}_j(t)\pi_j(t)p_j(t),\end{aligned}$$

with control variables $c_{jk}(t)$ for all $k \in [0,1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. Note that we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian. (To ease notation we now drop the time index t.)

We apply the necessary conditions for a maximum to the household's problem given by Acemoglu (2009, theorem 7.9). These conditions form the basis of the model's equilibrium conditions.

The first optimality conditions are $\partial\mathcal{H}_j/\partial c_{jk} = 0$ for all $k \in [0,1]$. They yield

$$\frac{1}{c_j}\left(\frac{c_{jk}}{c_j}\right)^{-1/\epsilon} = \mathcal{A}_j p_k. \tag{A1}$$

Appropriately integrating (A1) over all $k \in [0,1]$ and using the expressions for the consumption and price indices,

$$c_j(t) = \left[\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\,dk\right]^{\epsilon/(\epsilon-1)} \tag{A2}$$

$$p(t) = \left[\int_0^1 p_j(t)^{1-\epsilon}\,di\right]^{1/(1-\epsilon)}, \tag{A3}$$

we find

$$\mathcal{A}_j = \frac{1}{pc_j}. \tag{A4}$$

Moreover, combining (A1) and (A4), we obtain

$$c_{jk} = \left(\frac{p_k}{p}\right)^{-\epsilon} c_j. \tag{A5}$$

Integrating (A5) over all $j \in [0, 1]$, we get the usual demand for good k:

$$y_k^d(p_k) = \int_0^1 c_{jk}\, dj = \left(\frac{p_k}{p}\right)^{-\epsilon} c, \tag{A6}$$

where $c = \int_0^1 c_j\, dj$ is aggregate consumption. We use this expression for $y_k^d(p_k)$ in household k's Hamiltonian. Equation (A5) also implies that

$$\int_0^1 p_k c_{jk}\, dk = \int_0^1 p_k \left(\frac{p_k}{p}\right)^{-\epsilon} c_j\, dk = pc_j.$$

This means that when consumption expenditure is allocated optimally across goods, the price of one unit of consumption index is p.

The second optimality condition is $\partial\mathcal{H}_j/\partial b_j = \delta\mathcal{A}_j - \dot{\mathcal{A}}_j$, which gives

$$-\frac{\dot{\mathcal{A}}_j}{\mathcal{A}_j} = i^h + \frac{1}{p\mathcal{A}_j} \cdot u'\left(\frac{b_j - b}{p}\right) - \delta.$$

Using (A4) and $i^h = i + \sigma$, we obtain the household's Euler equation:

$$\frac{\dot{c}_j}{c_j} = i + \sigma - \pi + c_j u'\left(\frac{b_j - b}{p}\right) - \delta. \tag{A7}$$

This equation describes the optimal path for household j's consumption.

The third optimality condition is $\partial\mathcal{H}_j/\partial\pi_j = 0$, which yields

$$\mathcal{B}_j p_j = \gamma\pi_j. \tag{A8}$$

Differentiating (A8) with respect to time, we obtain

$$\frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j. \tag{A9}$$

The last optimality condition is $\partial\mathcal{H}_j/\partial p_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j$, which implies

$$\frac{\kappa}{a} \cdot \frac{\epsilon y_j}{p_j} - (\epsilon - 1)\mathcal{A}_j y_j + \mathcal{B}_j\pi_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j.$$

Reshuffling the terms then yields

$$\pi_j - \frac{(\epsilon - 1) y_j \mathcal{A}_j}{\mathcal{B}_j p_j} \left(p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa}{a \mathcal{A}_j} \right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Finally, incorporating (A4), (A8), and (A9), we obtain the household's Phillips curve:

$$\text{(A10)} \qquad \frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j} \left(\frac{p_j}{p} - \frac{\epsilon}{\epsilon - 1} \cdot \frac{\kappa c_j}{a} \right).$$

This equation describes the optimal path for the price set by household j.

A.2. Equilibrium

We now describe the equilibrium of the model. Since all households face the same initial conditions, they all behave the same. We therefore drop the subscripts j and k on all the variables. In particular, all households hold the same wealth, so relative wealth is zero: $b_j = b$. In addition, production and consumption are equal in equilibrium: $y = c$.

Accordingly, the household's Phillips curve, given by (A10), simplifies to

$$\dot{\pi} = \delta \pi - \frac{\epsilon \kappa}{\gamma a} (y - y^n),$$

where

$$\text{(A11)} \qquad y^n = \frac{\epsilon - 1}{\epsilon} \cdot \frac{a}{\kappa}.$$

And the household's Euler equation, given by (A7), simplifies to

$$\frac{\dot{y}}{y} = r - r^n + u'(0)(y - y^n),$$

where $r = i - \pi$ and

$$\text{(A12)} \qquad r^n = \delta - \sigma - u'(0) y^n.$$

These differential equations are the Phillips curve (1) and Euler equation (4).

Appendix B. Heuristic derivation of Euler equation & Phillips curve

To better understand and interpret the continuous-time Euler equation and Phillips curve, we complement the formal derivations of online appendix A with heuristic derivations, as in Blanchard and Fischer (1989, pp. 40–42).

B.1. Euler equation

The Euler equation says that households save in an optimal fashion: they cannot improve their situation by shifting consumption a little bit across time.

Consider a household delaying consumption of one unit of output from time t to time $t+dt$. The unit of output, invested at a real interest rate $r^h(t)$, becomes $1+r^h(t)dt$ at time $t+dt$. Given log consumption utility, the marginal utility from consumption at any time t is $e^{-\delta t}/y(t)$. Hence, the household forgoes $e^{-\delta t}/y(t)$ utils at time t and gains

$$[1+r^h(t)dt]\frac{e^{-\delta(t+dt)}}{y(t+dt)}$$

utils at time $t+dt$.

Since people enjoy holding wealth, the one unit of output saved between t and $t+dt$ provides hedonic returns in addition to financial returns. The marginal utility from real wealth at time t is $e^{-\delta t}u'(0)$. Hence, by holding an extra unit of real wealth for a duration dt, the household gains $e^{-\delta t}u'(0)dt$ utils.

At the optimum, reallocating consumption over time does not affect utility, so the following holds:

$$0=-\frac{e^{-\delta t}}{y(t)}+\left[1+r^h(t)dt\right]\frac{e^{-\delta(t+dt)}}{y(t+dt)}+e^{-\delta t}u'(0)dt.$$

Divided by $e^{-\delta t}/y(t)$, this condition becomes

$$1=[1+r^h(t)dt]e^{-\delta dt}\frac{y(t)}{y(t+dt)}+u'(0)y(t)dt.$$

Furthermore, up to second-order terms, the following approximations are valid:

$$\begin{aligned}
e^{-\delta dt}&=1-\delta dt\\
\frac{y(t+dt)}{y(t)}&=1+\frac{\dot{y}(t)}{y(t)}dt\\
\frac{1}{1+xdt}&=1-xdt,\quad \text{for any } x.
\end{aligned}$$

Hence, up to second-order terms, the previous condition gives

$$1 = \left[1 + r^h(t)dt\right](1 - \delta dt)\left[1 - \frac{\dot{y}(t)}{y(t)}dt\right] + u'(0)y(t)dt.$$

Keeping only first-order terms, we obtain

$$1 = 1 - \delta dt + r^h(t)dt - \frac{\dot{y}(t)}{y(t)}dt + u'(0)y(t)dt.$$

Reshuffling the terms and dividing by dt, we conclude that

$$\frac{\dot{y}(t)}{y(t)} = r^h(t) - \delta + u'(0)y(t).$$

We obtain the Euler equation (4) from here by noting that $r^h(t) = r(t) + \sigma$ and introducing the natural rate of interest r^n given by (A12).

B.2. Phillips curve

The Phillips curve says that households price in an optimal fashion: they cannot improve their situation by shifting inflation a little bit across time.

Consider a household delaying one percentage point of inflation from time t to time $t + dt$. Given the quadratic price-change disutility, the marginal disutility from inflation at any time t is $e^{-\delta t}\gamma\pi(t)$. Hence, at time t, the household avoids a disutility of

$$e^{-\delta t}\gamma\pi(t) \times 1\%.$$

And, at time $t + dt$, the household incurs an extra disutility of

$$e^{-\delta(t+dt)}\gamma\pi(t + dt) \times 1\%.$$

Delaying inflation by one percentage point reduces the household's price between times t and $t + dt$ by $dp(t) = -1\% \times p(t)$. The price drop then affects sales. Since the price elasticity of demand is $-\epsilon$, sales increase by

$$dy(t) = -\epsilon y(t) \times -1\% = \epsilon y(t) \times 1\%.$$

Accordingly, the household's revenue grows by

$$d(p(t)y(t)) = p(t)dy(t) + y(t)dp(t) = (\epsilon - 1)y(t)p(t) \times 1\%.$$

With a higher revenue, the household can afford to consume more. Since in equilibrium all prices are the same, equal to $p(t)$, the increase in revenue raises consumption by

$$dc(t) = \frac{d(p(t)y(t))}{p(t)} = (\epsilon - 1)y(t) \times 1\%.$$

Hence, between times t and $t + dt$, the utility of consumption increases by

$$\frac{e^{-\delta t}}{y(t)}dc(t) = e^{-\delta t}(\epsilon - 1) \times 1\%.$$

At the same time, because production is higher, the household must work more. Hours worked are extended by

$$dh(t) = \frac{dy(t)}{a} = \frac{\epsilon y(t)}{a} \times 1\%.$$

As a result, between times t and $t + dt$, the disutility of labor is elevated by

$$e^{-\delta t}\kappa dh(t) = e^{-\delta t}\frac{\kappa\epsilon y(t)}{a} \times 1\%.$$

At the optimum, shifting inflation across time does not affect utility, so the following holds:

$$0 = e^{-\delta t}\gamma\pi(t) \times 1\% - e^{-\delta(t+dt)}\gamma\pi(t + dt) \times 1\% + e^{-\delta t}(\epsilon - 1) \times 1\% \times dt - e^{-\delta t}\kappa\epsilon\frac{y(t)}{a} \times 1\% \times dt.$$

Divided by $e^{-\delta t} \times 1\%$, this condition yields

$$0 = \gamma\pi(t) - e^{-\delta dt}\gamma\pi(t + dt) + (\epsilon - 1) \times dt - \kappa\epsilon\frac{y(t)}{a} \times dt.$$

Furthermore, up to second-order terms, the following approximations hold:

$$e^{-\delta dt} = 1 - \delta dt$$
$$\pi(t + dt) = \pi(t) + \dot{\pi}(t)dt.$$

Therefore, up to second-order terms, the previous condition gives

$$0 = \gamma\pi(t) - (1 - \delta dt)\gamma\left[\pi(t) + \dot{\pi}(t)dt\right] - \kappa\epsilon\frac{y(t)}{a}dt + (\epsilon - 1)dt.$$

Then, keeping only first-order terms, we obtain

$$0 = \delta\gamma\pi(t)dt - \gamma\dot{\pi}(t)dt - \kappa\epsilon\frac{y(t)}{a}dt + (\epsilon - 1)dt.$$

Rearranging the terms and dividing by γdt, we conclude that

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon\kappa}{\gamma a}\left[y(t) - \frac{\epsilon - 1}{\epsilon}\cdot\frac{a}{\kappa}\right].$$

Once we introduce the natural level of output y^n given by (A11), we obtain the Phillips curve (1).

The Phillips curve implies that without price-adjustment cost ($\gamma = 0$), households would produce at the natural level of output. This result comes from the monopolistic nature of competition. Without price-adjustment cost, it is optimal to charge a relative price that is a markup $\epsilon/(\epsilon - 1)$ over the real marginal cost. In turn, the real marginal cost is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. In equilibrium, all relative prices are 1, the marginal rate of substitution between labor and consumption is $\kappa/(1/y) = \kappa y$, and the marginal product of labor is a. Hence, optimal pricing requires

$$1 = \frac{\epsilon}{\epsilon - 1}\cdot\frac{\kappa y}{a}.$$

Combined with (A11), this condition implies $y = y^n$.

The derivation also elucidates why in steady state, inflation is positive whenever output is above its natural level. When inflation is positive, a household can reduce its price-adjustment cost by lowering its inflation. Since pricing is optimal, however, there cannot exist any profitable deviation from the equilibrium. This means that the household must also incur a cost when it lowers inflation. A consequence of lowering inflation is that the price charged by the household drops, which stimulates its sales and production. The absence of profitable deviation imposes that the household incurs a cost when production increases. In other words, production must be excessive: output must be above its natural level.

Appendix C. Euler equation & Phillips curve in discrete time

We recast the model of section 3 in discrete time, and we rederive the Euler equation and Phillips curve. This reformulation might be helpful to compare our model to the textbook New Keynesian model, which is presented in discrete time (Woodford 2003; Gali 2008). The reformulation also shows that introducing wealth in the utility function yields a discounted Euler equation.

C.1. Assumptions

The assumptions are the same in the discrete-time model as in the continuous-time model, except for government bonds. In discrete time, households trade one-period government bonds. Bonds purchased in period t have a price $q(t)$ and pay one unit of money in period $t+1$. The nominal interest rate on government bonds is defined as $i^h(t) = -\ln(q(t))$.

C.2. Household's problem

Household j chooses sequences $\left\{y_j(t), p_j(t), h_j(t), \left[c_{jk}(t)\right]_{k=0}^{1}, b_j(t)\right\}_{t=0}^{\infty}$ to maximize the discounted sum of instantaneous utilities

$$\sum_{t=0}^{\infty} \beta^t \left\{\frac{\epsilon}{\epsilon-1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right) + u\left(\frac{b_j(t)-b(t)}{p(t)}\right) - \kappa h_j(t) - \frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)} - 1\right]^2\right\} dt,$$

where $\beta < 1$ is the time discount factor. The maximization is subject to three constraints. First, there is a production function: $y_j(t) = a h_j(t)$. Second, there is the demand for good j:

$$y_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t) \equiv y_j^d(p_j(t), t).$$

The demand for good j is the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time. And third, there is a budget constraint:

$$\int_0^1 p_k(t) c_{jk}(t)\, dk + q(t) b_j(t) + \tau(t) = p_j(t) y_j(t) + b_j(t-1).$$

Household j is also subject to a solvency constraint preventing Ponzi schemes. Lastly, household j takes as given the initial conditions $b_j(-1)$ and $p_j(-1)$, as well as the sequences of aggregate variables $\{p(t), q(t), c(t)\}_{t=0}^{\infty}$.

The Lagrangian of the household's problem is

$$\mathcal{L}_j = \sum_{t=0}^{\infty} \beta^t \Bigg\{ \frac{\epsilon}{\epsilon-1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk \right) + u\left(\frac{b_j(t)-b(t)}{p(t)} \right) - \frac{\kappa}{a} y_j^d(p_j(t),t) - \frac{\gamma}{2} \left[\frac{p_j(t)}{p_j(t-1)} - 1 \right]^2$$
$$+ \mathcal{A}_j(t) \left[p_j(t) y_j^d(p_j(t),t) + b_j(t-1) - \int_0^1 p_k(t) c_{jk}(t)\, dk - q(t) b_j(t) - \tau(t) \right] \Bigg\},$$

where $\mathcal{A}_j(t)$ is a Lagrange multiplier. We have used the production and demand constraints to substitute $h_j(t)$ and $y_j(t)$ out of the Lagrangian.

The necessary conditions for a maximum to the household's problem are standard first-order conditions. The first optimality conditions are $\partial \mathcal{L}_j / \partial c_{jk}(t) = 0$ for all $k \in [0,1]$ and all t. As in continuous time, these conditions yield

$$\mathcal{A}_j(t) = \frac{1}{p(t) c_j(t)}. \tag{A13}$$

The second optimality condition is $\partial \mathcal{L}_j / \partial b_j(t) = 0$ for all t, which gives

$$q(t)\mathcal{A}_j(t) = \frac{1}{p(t)} u'\left(\frac{b_j(t)-b(t)}{p(t)} \right) + \beta \mathcal{A}_j(t+1).$$

Using (A13), we obtain the household's Euler equation:

$$q(t) = c_j(t) u'\left(\frac{b_j(t)-b(t)}{p(t)} \right) + \beta \frac{p(t) c_j(t)}{p(t+1) c_j(t+1)}. \tag{A14}$$

The third optimality condition is $\partial \mathcal{L}_j / \partial p_j(t) = 0$ for all t, which yields

$$0 = \frac{\kappa}{a} \cdot \frac{\epsilon y_j(t)}{p_j(t)} - \frac{\gamma}{p_j(t-1)} \left[\frac{p_j(t)}{p_j(t-1)} - 1 \right] + (1-\epsilon) \mathcal{A}_j(t) y_j(t) + \beta \gamma \frac{p_j(t+1)}{p_j(t)^2} \left[\frac{p_j(t+1)}{p_j(t)} - 1 \right].$$

Multiplying this equation by $p_j(t)/\gamma$ and using (A13), we obtain the household's Phillips curve:

$$\frac{p_j(t)}{p_j(t-1)} \left[\frac{p_j(t)}{p_j(t-1)} - 1 \right] = \beta \frac{p_j(t+1)}{p_j(t)} \left[\frac{p_j(t+1)}{p_j(t)} - 1 \right] + \frac{\epsilon \kappa}{\gamma a} y_j(t) - \frac{\epsilon - 1}{\gamma} \cdot \frac{p_j(t) y_j(t)}{p(t) c_j(t)}. \tag{A15}$$

C.3. Equilibrium

We now describe the equilibrium. Since all households face the same initial conditions, they all behave the same, so we drop the subscripts j and k on all the variables. In particular, all households hold the same wealth, so relative wealth is zero: $b_j(t) = b(t)$. In addition, production and consumption are equal in equilibrium: $y(t) = c(t)$.

Accordingly, from (A14) we obtain the Euler equation

$$q(t) = u'(0)y(t) + \beta \frac{p(t)y(t)}{p(t+1)y(t+1)}. \tag{A16}$$

Moreover, combining (A15) and (A11), we obtain the Phillips curve

$$\frac{p(t)}{p(t-1)}\left[\frac{p(t)}{p(t-1)} - 1\right] = \beta\frac{p(t+1)}{p(t)}\left[\frac{p(t+1)}{p(t)} - 1\right] + \frac{\epsilon - 1}{\gamma}\left[\frac{y(t)}{y^n} - 1\right]. \tag{A17}$$

C.4. Log-linearization

To obtain the standard expressions of the Euler equation and Phillips curve, we log-linearize (A16) and (A17) around the natural steady state: where $y = y^n$, $\pi = 0$, and $i = r^n$. To that end, we introduce the log-deviation of output from its steady-state level: $\hat{y}(t) = \ln(y(t)) - \ln(y^n)$. We also introduce the inflation rate between periods t and $t+1$: $\pi(t+1) = \ln(p(t+1)) - \ln(p(t))$.

Euler equation. We start by log-linearizing the Euler equation (A16).

We first take the log of the left-hand side of (A16). Using the discrete-time definition of the nominal interest rate faced by households, $i^h(t)$, we obtain $\ln(q(t)) = -i^h(t)$. At the natural steady state, the monetary-policy rate is $i = r^n$, so the interest rate faced by households is $i^h = r^n + \sigma$, and $\ln(q(t)) = -r^n - \sigma$.

Next we take the log of the right-hand side of (A16). We obtain $\Lambda \equiv \ln(\Lambda_1 + \Lambda_2)$, where

$$\Lambda_1 \equiv u'(0)y(t), \qquad \Lambda_2 \equiv \beta\frac{p(t)y(t)}{p(t+1)y(t+1)}.$$

For future reference, we compute the values of Λ, Λ_1, and Λ_2 at the natural steady state. At the natural steady state, the log of the left-hand side of (A16) equals $-r^n - \sigma$, which implies that the log of the right-hand side of (A16) must also equal $-r^n - \sigma$. That is, at the natural steady state, $\Lambda = -r^n - \sigma$. Moreover, at that steady state, $\Lambda_1 = u'(0)y^n$. And, since inflation is zero and output is constant at that steady state, $\Lambda_2 = \beta$.

Using these results, we obtain a first-order approximation of $\Lambda(\Lambda_1, \Lambda_2)$ around the natural steady state:

$$\Lambda = -r^n - \sigma + \frac{\partial\Lambda}{\partial\Lambda_1}\left[\Lambda_1 - u'(0)y^n\right] + \frac{\partial\Lambda}{\partial\Lambda_2}\left[\Lambda_2 - \beta\right].$$

Factoring out $u'(0)\, y^n$ and β, and using the definitions of Λ_1 and Λ_2, we obtain

$$\Lambda = -r^n - \sigma + u'(0)\, y^n \cdot \frac{\partial \Lambda}{\partial \Lambda_1} \cdot \left[\frac{y(t)}{y^n} - 1\right] + \beta \cdot \frac{\partial \Lambda}{\partial \Lambda_2} \cdot \left[\frac{p(t)\, y(t)}{p(t+1)\, y(t+1)} - 1\right]. \tag{A18}$$

Since $\Lambda = \ln(\Lambda_1 + \Lambda_2)$, we obviously have

$$\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{\Lambda_1 + \Lambda_2}.$$

In (A18), the derivatives are evaluated at the natural state, so

$$\frac{\partial \Lambda}{\partial \Lambda_1} = \frac{\partial \Lambda}{\partial \Lambda_2} = \frac{1}{u'(0)\, y^n + \beta}.$$

Hence, (A18) becomes

$$\Lambda = -r^n - \sigma + \frac{u'(0)\, y^n}{u'(0)\, y^n + \beta} \left[\frac{y(t)}{y^n} - 1\right] + \frac{\beta}{u'(0)\, y^n + \beta} \left[\frac{p(t)\, y(t)}{p(t+1)\, y(t+1)} - 1\right]. \tag{A19}$$

Last, up to second-order terms, we have $\ln(x) = x - 1$ around $x = 1$. Thus, we have the following first-order approximations around the natural steady state:

$$\frac{y(t)}{y^n} - 1 = \ln\left(\frac{y(t)}{y^n}\right) = \hat{y}(t) \tag{A20}$$

and

$$\begin{aligned} \frac{p(t)\, y(t)}{p(t+1)\, y(t+1)} - 1 &= \ln\left(\frac{p(t)\, y(t)}{p(t+1)\, y(t+1)}\right) \\ &= \ln\left(\frac{y(t)}{y^n}\right) - \ln\left(\frac{y(t+1)}{y^n}\right) - \ln\left(\frac{p(t+1)}{p(t)}\right) \\ &= \hat{y}(t) - \hat{y}(t+1) - \pi(t+1). \end{aligned}$$

We therefore rewrite (A19) as

$$\Lambda = -r^n - \sigma + \frac{u'(0)\, y^n}{u'(0)\, y^n + \beta} \hat{y}(t) + \frac{\beta}{u'(0)\, y^n + \beta} \left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right].$$

Finally, introducing

$$\alpha = \frac{\beta}{\beta + u'(0)\, y^n},$$

we obtain

$$\Lambda = -r^n - \sigma + (1-\alpha)\hat{y}(t) + \alpha\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right].$$

In conclusion, taking the log of the Euler equation (A16) yields

$$-i^h(t) = -r^n - \sigma + (1-\alpha)\hat{y}(t) + \alpha\left[\hat{y}(t) - \hat{y}(t+1) - \pi(t+1)\right].$$

Reshuffling the terms and noting that $i^h(t) = i(t) + \sigma$, we obtain the log-linearized Euler equation:

$$\hat{y}(t) = \alpha\hat{y}(t+1) - \left[i(t) - r^n - \alpha\pi(t+1)\right]. \tag{A21}$$

Discounting. Because $u'(0) > 0$, we have

$$\alpha = \frac{\beta}{\beta + u'(0)y^n} < 1.$$

Thus, because the marginal utility of wealth is positive, the Euler equation is discounted: future output, $\hat{y}(t+1)$, appears discounted by the coefficient $\alpha < 1$ in (A21). Such discounting also appears in the presence of overlapping generations (Del Negro, Giannoni, and Patterson 2015; Eggertsson, Mehrotra, and Robbins 2019); heterogeneous agents facing borrowing constraints and cyclical income risk (McKay, Nakamura, and Steinsson 2017; Acharya and Dogra 2020; Bilbiie 2019); consumers' bounded rationality (Gabaix 2020); incomplete information (Angeletos and Lian 2018); bonds in the utility function (Campbell et al. 2017); and borrowing costs increasing in household debt (Beaudry and Portier 2018).

To make discounting more apparent, we solve the Euler equation forward:

$$\hat{y}(t) = -\sum_{k=0}^{+\infty}\alpha^k\left[i(t+k) - r^n - \alpha\pi(t+k+1)\right].$$

The effect on current output of interest rates k periods in the future is discounted by $\alpha^k < 1$; hence, discounting is stronger for interest rates further in the future (McKay, Nakamura, and Steinsson 2017, p. 821).

Phillips curve. Next we log-linearize the Phillips curve (A17).

We start with the left-hand side of (A17). The first-order approximations of $x(x-1)$ and $\ln(x)$ around $x = 1$ both are $x - 1$. This means that up to second-order terms, we have $x(x-1) = \ln(x)$ around $x = 1$. Hence, up to second-order terms, the following approximation holds around the natural steady state:

$$\frac{p(t)}{p(t-1)}\left[\frac{p(t)}{p(t-1)}-1\right] = \ln\left(\frac{p(t)}{p(t-1)}\right) = \pi(t).$$

We turn to the right-hand side of (A17) and proceed similarly. We find that up to second-order terms, the following approximation holds around the natural steady state:

$$\beta\frac{p(t+1)}{p(t)}\left[\frac{p(t+1)}{p(t)}-1\right] = \beta\ln\left(\frac{p(t+1)}{p(t)}\right) = \beta\pi(t+1).$$

Furthermore, (A20) implies that up to second-order terms, the ensuing approximation holds around the natural steady state:

$$\frac{\epsilon-1}{\gamma}\left[\frac{y(t)}{y^n}-1\right] = \frac{\epsilon-1}{\gamma}\hat{y}(t).$$

Combining all these results, we obtain the log-linearized Phillips curve:

$$\pi(t) = \beta\pi(t+1) + \frac{\epsilon-1}{\gamma}\hat{y}(t). \tag{A22}$$

Appendix D. Proofs

We provide alternative proofs of propositions 1 and 2. These proofs are not graphical but algebraic; they are closer to the proofs found in the literature. We also complement the proof of proposition 4.

D.1. Alternative proof of proposition 1

We study the properties of the dynamical system generated by the Phillips curve (1) and Euler equation (4) in normal times. The natural rate of interest is positive and monetary policy imposes $r(\pi) = r^n + (\phi - 1)\pi$.

Steady state. A steady state $[y, \pi]$ must satisfy the steady-state Phillips curve (3) and steady-state Euler equation (7). These equations form a linear system:

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}(y - y^n)$$
$$(\phi - 1)\pi = -u'(0)(y - y^n).$$

As $[y = y^n, \pi = 0]$ satisfies both equations, it is a steady state. Furthermore the steady state is unique because the two equations are non-parallel. In the NK model, this is obvious since $u'(0) = 0$. In the WUNK model, the slope of the second equation is $-u'(0)/(\phi - 1)$. If $\phi > 1$, the slope is negative. If $\phi \in [0, 1)$, the slope is positive and strictly greater than $u'(0)$ and thus than $\epsilon\kappa/(\delta\gamma a)$ (because (9) holds). In both cases, the two equations have different slopes.

Linearization. The Euler-Phillips system is nonlinear, so we determine its properties by linearizing it around its steady state. We first write the Euler equation and Phillips curve as

$$\dot{y}(t) = E(y(t), \pi(t)), \quad \text{where } E(y, \pi) = y[(\phi - 1)\pi + u'(0)(y - y^n)]$$
$$\dot{\pi}(t) = P(y(t), \pi(t)), \quad \text{where } P(y, \pi) = \delta\pi - \frac{\epsilon\kappa}{\gamma a}(y - y^n).$$

Around the natural steady state, the linearized Euler-Phillips system is

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial E}{\partial y} & \frac{\partial E}{\partial \pi} \\ \frac{\partial P}{\partial y} & \frac{\partial P}{\partial \pi} \end{bmatrix} \begin{bmatrix} y(t) - y^n \\ \pi \end{bmatrix},$$

where the derivatives are evaluated at $[y = y^n, \pi = 0]$. We have

$$\frac{\partial E}{\partial y} = y^n u'(0), \qquad \frac{\partial E}{\partial \pi} = y^n(\phi - 1)$$
$$\frac{\partial P}{\partial y} = -\frac{\epsilon\kappa}{\gamma a}, \qquad \frac{\partial P}{\partial \pi} = \delta.$$

Accordingly the linearized Euler-Phillips system is

$$\begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} u'(0)y^n & (\phi - 1)y^n \\ -\epsilon\kappa/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix}. \tag{A23}$$

We denote by $\boldsymbol{M}$ the matrix in (A23), and by $\mu_1 \in \mathbb{C}$ and $\mu_2 \in \mathbb{C}$ the two eigenvalues of $\boldsymbol{M}$, assumed to be distinct.

Solution with two real eigenvalues. We begin by solving (A23) when μ_1 and μ_2 are real and nonzero. Without loss of generality, we assume $\mu_1 < \mu_2$. Then the solution takes the form

$$\begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix} = x_1 e^{\mu_1 t} \boldsymbol{v}_1 + x_2 e^{\mu_2 t} \boldsymbol{v}_2, \tag{A24}$$

where $\boldsymbol{v}_1 \in \mathbb{R}^2$ and $\boldsymbol{v}_2 \in \mathbb{R}^2$ are the linearly independent eigenvectors respectively associated with the eigenvalues μ_1 and μ_2, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, p. 35).

From (A24), we see that the Euler-Phillips system is a source when $\mu_1 > 0$ and $\mu_2 > 0$. Moreover, the solutions are tangent to $\boldsymbol{v}_1$ when $t \to -\infty$ and are parallel to $\boldsymbol{v}_2$ when $t \to +\infty$. The system is a saddle when $\mu_1 < 0$ and $\mu_2 > 0$; in that case, the vector $\boldsymbol{v}_1$ gives the direction of the stable line (saddle path) while the vector $\boldsymbol{v}_2$ gives the direction of the unstable line. Lastly, when $\mu_1 < 0$ and $\mu_2 < 0$, the system is a sink. (See Hirsch, Smale, and Devaney 2013, pp. 40–44.)

Solution with two complex eigenvalues. Next we solve (A23) when μ_1 and μ_2 are complex conjugates. We write the eigenvalues as $\mu_1 = \theta + i\varsigma$ and $\mu_2 = \theta - i\varsigma$. We also write the eigenvector associated with μ_1 as $\boldsymbol{v}_1 + i\boldsymbol{v}_2$, where the vectors $\boldsymbol{v}_1 \in \mathbb{R}^2$ and $\boldsymbol{v}_2 \in \mathbb{R}^2$ are linearly independent. Then the solution takes a more complicated form:

$$\begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix} = e^{\theta t} \left[\boldsymbol{v}_1, \boldsymbol{v}_2\right] \begin{bmatrix} \cos(\varsigma t) & \sin(\varsigma t) \\ -\sin(\varsigma t) & \cos(\varsigma t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $[\boldsymbol{v}_1, \boldsymbol{v}_2] \in \mathbb{R}^{2\times 2}$ is a 2×2 matrix, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition (Hirsch, Smale, and Devaney 2013, pp. 44–55).

These solutions wind periodically around the steady state, either moving toward it ($\theta < 0$) or away from it ($\theta > 0$). Hence, the Euler-Phillips system is a spiral source if $\theta > 0$ and a spiral sink if $\theta < 0$. In the special case $\theta = 0$, the solutions circle around the steady state: the Euler-Phillips system is a center. (See Hirsch, Smale, and Devaney 2013, pp. 44–47.)

Classification. We classify the Euler-Phillips system from the trace and determinant of $\boldsymbol{M}$ (Hirsch, Smale, and Devaney 2013, pp. 61–64). The classification relies on the property that $\operatorname{tr}(\boldsymbol{M}) = \mu_1 + \mu_2$ and $\det(\boldsymbol{M}) = \mu_1\mu_2$. The following situations may occur in the NK and WUNK models:

- $\det(\boldsymbol{M}) < 0$: Then the Euler-Phillips system is a saddle. This is because $\det(\boldsymbol{M}) < 0$ indicates that μ_1 and μ_2 are real, nonzero, and of opposite sign. Indeed, if μ_1 and μ_2 were real and of the same sign, $\det(\boldsymbol{M}) = \mu_1\mu_2 > 0$; and if they were complex conjugates, $\det(\boldsymbol{M}) = \mu_1\overline{\mu_1} = \operatorname{Re}(\mu_1)^2 + \operatorname{Im}(\mu_1)^2 > 0$.

- $\det(\boldsymbol{M}) > 0$ and $\operatorname{tr}(\boldsymbol{M}) > 0$: Then the Euler-Phillips system is a source. This is because $\det(\boldsymbol{M}) > 0$ indicates that μ_1 and μ_2 are either real, nonzero, and of the same sign; or complex conjugates. Since in addition $\operatorname{tr}(\boldsymbol{M}) > 0$, μ_1 and μ_2 must be either real and positive, or complex with a positive real part. Indeed, if μ_1 and μ_2 were real and negative, $\operatorname{tr}(\boldsymbol{M}) = \mu_1 + \mu_2 < 0$; if they were complex with a negative real part, $\operatorname{tr}(\boldsymbol{M}) = \mu_1 + \overline{\mu_1} = 2\operatorname{Re}(\mu_1) < 0$.

Using (A23), we compute the trace and determinant of $\boldsymbol{M}$:

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{M}) &= \delta + u'(0)y^n \\
\det(\boldsymbol{M}) &= \delta u'(0)y^n + (\phi - 1)\frac{\epsilon\kappa}{\gamma a}y^n.
\end{aligned}$$

In the NK model, $u'(0) = 0$, so $\operatorname{tr}(\boldsymbol{M}) = \delta > 0$ and

$$\det(\boldsymbol{M}) = (\phi - 1)\frac{y^n\epsilon\kappa}{\gamma a}.$$

If $\phi > 1$, $\operatorname{tr}(\boldsymbol{M}) > 0$ and $\det(\boldsymbol{M}) > 0$, so the system is a source. If $\phi < 1$, $\det(\boldsymbol{M}) < 0$, so the system is a saddle.

In the WUNK model, $\operatorname{tr}(\boldsymbol{M}) > \delta > 0$. Further, using $\phi - 1 \geq -1$ and (9), we have

$$\det(\boldsymbol{M}) \geq \delta u'(0)y^n - \frac{\epsilon\kappa}{\gamma a}y^n = \delta y^n\left[u'(0) - \frac{\epsilon\kappa}{\delta\gamma a}\right] > 0.$$

Since $\mathrm{tr}(\boldsymbol{M}) > 0$ and $\det(\boldsymbol{M}) > 0$, the system is a source.

D.2. Alternative proof of proposition 2

We study the properties of the dynamical system generated by the Phillips curve (1) and Euler equation (4) at the ZLB. The natural rate of interest is negative and monetary policy imposes $r(\pi) = -\pi$.

Steady state. A steady state $[y, \pi]$ must satisfy the steady-state Phillips curve (3) and the steady-state Euler equation (7). These equations form a linear system:

$$\pi = \frac{\epsilon\kappa}{\delta\gamma a}(y - y^n) \tag{A25}$$

$$\pi = -r^n + u'(0)(y - y^n). \tag{A26}$$

A solution to this system with positive output is a steady state.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

$$\pi^z = -r^n$$
$$y^z = y^n - \frac{\delta\gamma a}{\epsilon\kappa} r^n.$$

Since $r^n < 0$, the solution satisfies $y^z > y^n > 0$: the solution has positive output so it is a steady state. Hence the NK model admits a unique steady state at the ZLB: $[y^z, \pi^z]$, where $\pi^z > 0$ (since $r^n < 0$) and $y^z > y^n$.

In the WUNK model, since (9) holds, the equations (A25) and (A26) are non-parallel, so the system admits a unique solution, denoted $[y^z, \pi^z]$. Using (A25) to substitute $y - y^n$ out of (A26), we find that

$$\pi^z = \frac{r^n}{u'(0)\delta\gamma a/(\epsilon\kappa) - 1}. \tag{A27}$$

Condition (9) implies that the denominator is positive. Since $r^n < 0$, we conclude that $\pi^z < 0$.

Next, combining (A25) and (A27), we obtain

$$y^z = y^n + \frac{r^n}{u'(0) - \epsilon\kappa/(\delta\gamma a)}. \tag{A28}$$

Since (9) holds, the denominator of the fraction is positive. As $r^n < 0$, we conclude that $y^z < y^n$.

Finally, to establish that $[y^z, \pi^z]$ is a steady state, we need to verify that $y^z > 0$. According to

(A28), we need

$$y^n > \frac{-r^n}{u'(0) - \epsilon\kappa/(\delta\gamma a)}.$$

Equations (5) and (9) indicate that

$$-r^n = u'(0)y^n + \sigma - \delta \quad \text{and} \quad u'(0) - \frac{\epsilon\kappa}{\delta\gamma a} > 0.$$

The above inequality is therefore equivalent to

$$\left[u'(0) - \frac{\epsilon\kappa}{\delta\gamma a}\right] y^n > u'(0)y^n + \sigma - \delta.$$

Reshuffling terms, we rewrite the inequality as

$$\delta > \sigma + \frac{\epsilon\kappa y^n}{\delta\gamma a}.$$

Equation (A11) implies that

$$\frac{\epsilon\kappa y^n}{\gamma a} = \frac{\epsilon - 1}{\gamma}.$$

So we need to verify that

$$\delta > \sigma + \frac{\epsilon - 1}{\delta\gamma}.$$

But we have imposed this condition in the WUNK model, to accommodate positive natural rates of interest. We therefore conclude that $y^z > 0$ and that $[y^z, \pi^z]$ is a steady state.

Linearization. The Euler-Phillips system is nonlinear, so we determine its properties by linearizing it. Around the ZLB steady state, the linearized Euler-Phillips system is

$$\text{(A29)} \qquad \begin{bmatrix} \dot{y}(t) \\ \dot{\pi}(t) \end{bmatrix} = \begin{bmatrix} u'(0)y^z & -y^z \\ -\epsilon\kappa/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y(t) - y^z \\ \pi(t) - \pi^z \end{bmatrix}.$$

To obtain the matrix, denoted $\boldsymbol{M}$, we set $\phi = 0$ and replace y^n by y^z in the matrix from (A23).

Classification. We classify the Euler-Phillips system (A29) by computing the trace and determinant of $\boldsymbol{M}$, as in online appendix D.1. We have $\mathrm{tr}(\boldsymbol{M}) = \delta + u'(0)y^z > 0$ and

$$\det(\boldsymbol{M}) = \delta y^z \left[u'(0) - \frac{\epsilon\kappa}{\delta\gamma a}\right].$$

In the NK model, $u'(0) = 0$ so $\det(\boldsymbol{M}) < 0$, which implies that the Euler-Phillips system is a saddle. In the WUNK model, (9) implies that $\det(\boldsymbol{M}) > 0$. Since in addition $\mathrm{tr}(\boldsymbol{M}) > 0$, the Euler-Phillips system is a source. In fact, in the WUNK model, the discriminant of the characteristic equation of $\boldsymbol{M}$ is strictly positive:

$$\begin{aligned}\mathrm{tr}(\boldsymbol{M})^2 - 4\det(\boldsymbol{M}) &= \delta^2 + [u'(0)y^n]^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{\epsilon\kappa}{\gamma a}y^n \\ &= [\delta - u'(0)y^n]^2 + 4\frac{\epsilon\kappa}{\gamma a}y^n > 0.\end{aligned}$$

Hence the eigenvalues of $\boldsymbol{M}$ are real, not complex: the Euler-Phillips system is a nodal source, not a spiral source.

D.3. Complement to the proof of proposition 4

We characterize the forward-guidance duration Δ^* for the NK model, and the ZLB duration T^* for the WUNK model.

In the NK model, Δ^* is the duration of forward guidance that brings the economy on the unstable line of the ZLB phase diagram at time T (figure 3C). With longer forward guidance ($\Delta > \Delta^*$), the economy is above the unstable line at time T, and so it is connected to trajectories that come from the northeast quadrant of the ZLB phase diagram (figure 3D). As a consequence, during ZLB and forward guidance, inflation is positive and output is above its natural level. Moreover, since the position of the economy at the end of the ZLB is unaffected by the duration of the ZLB, initial output and inflation become arbitrarily high as the ZLB duration of the ZLB approaches infinity.

In the WUNK model, for any forward-guidance duration, the economy at time T is bound to be in the right-hand triangle of figure 4D. All the points in that triangle are connected to trajectories that flow from the ZLB steady state, through the left-hand triangle of figure 4D. For any of these trajectories, initial inflation $\pi(0)$ converges from above to the ZLB steady state's inflation π^z as the ZLB duration T goes to infinity. Since $\pi^z < 0$, we infer that for each trajectory, there is a ZLB duration $\hat{T}$, such that for any $T > \hat{T}$, $\pi(0) < 0$. Furthermore, as showed in figure 4D, $y(0) < y^n$ whenever $\pi(0) < 0$. The ZLB duration T^* is constructed as $T^* = \max\left\{\hat{T}\right\}$. The maximum exists because the right-hand triangle is a closed and bounded subset of $\mathbb{R}^2$, so the set $\left\{\hat{T}\right\}$ is a closed and bounded subset of $\mathbb{R}$, which admits a maximum. We know that the set $\left\{\hat{T}\right\}$ is closed and bounded because the function that maps a position at time T to a threshold $\hat{T}$ is continuous.

Appendix E. Model with government spending

We introduce government spending into the model of section 3. We compute the model's Euler equation and Phillips curve, linearize them, and use the linearized equations to construct the model's phase diagrams.

E.1. Assumptions

We start from the model of section 3, and we assume that the government purchases a quantity $g_j(t)$ of each good $j \in [0,1]$. These quantities are aggregated into an index of public consumption

$$g(t) \equiv \left[\int_0^1 g_j(t)^{(\epsilon-1)/\epsilon}\, dj\right]^{\epsilon/(\epsilon-1)}. \tag{A30}$$

Public consumption $g(t)$ enters separately into households' utility functions. Government expenditure is financed with lump-sum taxation.

Additionally, we assume that the disutility of labor is not linear but convex. Household j incurs disutility

$$\frac{\kappa^{1+\eta}}{1+\eta} h_j(t)^{1+\eta}$$

from working, where $\eta > 0$ is the inverse of the Frisch elasticity. The utility function is altered to ensure that government spending affects inflation and private consumption.

E.2. Euler equation & Phillips curve

We derive the Euler equation and Phillips curve just as in online appendix A.

The only new step is to compute the government's spending on each good. At any time t, the government chooses the amount $g_j(t)$ of each good $j \in [0,1]$ to minimize the expenditure

$$\int_0^1 p_j(t) g_j(t)\, dj$$

subject to the constraint of providing an amount of public consumption g:

$$\left[\int_0^1 g_j(t)^{(\epsilon-1)/\epsilon}\, dj\right]^{\epsilon/(\epsilon-1)} = g(t).$$

To solve the government's problem at time t, we set up a Lagrangian:

$$\mathcal{L} = \int_0^1 p_j(t) g_j(t)\, dj + C \cdot \left\{ g - \left[\int_0^1 g_j(t)^{(\epsilon-1)/\epsilon}\, dj \right]^{\epsilon/(\epsilon-1)} \right\},$$

where C is a Lagrange multiplier. We then follow the same steps as in the derivation of (A6). The first-order conditions with respect to $g_j(t)$ for all $j \in [0,1]$ are $\partial\mathcal{L}/\partial g_j = 0$. These conditions imply

(A31) $$p_j(t) = C \cdot \left[\frac{g_j(t)}{g(t)} \right]^{-1/\epsilon}.$$

Appropriately integrating (A31) over all $j \in [0,1]$, and using (A3) and (A30), we find

(A32) $$C = p(t).$$

Lastly, combining (A31) and (A32), we obtain the government's demand for good j:

(A33) $$g_j(t) = \left[\frac{p_j(t)}{p(t)} \right]^{-\epsilon} g(t).$$

Next we solve household j's problem. We set up the current-value Hamiltonian:

$$\mathcal{H}_j = \frac{\epsilon}{\epsilon-1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk \right) + u\left(\frac{b_j(t) - b(t)}{p(t)} \right) - \frac{1}{1+\eta} \left[\frac{\kappa}{a} y_j^d(p_j(t), t) \right]^{1+\eta} - \frac{\gamma}{2} \pi_j(t)^2$$
$$+ \mathcal{A}_j(t) \left[i^h(t) b_j(t) + p_j(t) y_j^d(p_j(t), t) - \int_0^1 p_k(t) c_{jk}(t)\, dk - \tau(t) \right] + \mathcal{B}_j(t) \pi_j(t) p_j(t).$$

The terms featuring the consumption levels $c_{jk}(t)$ in the Hamiltonian are the same as in online appendix A.1, so the optimality conditions $\partial\mathcal{H}_j/\partial c_{jk} = 0$ remain the same. This implies that (A1), (A4), and (A5) remain valid. Adding the government's demand, given by (A33), to households' demand, given by (A5), we obtain the total demand for good j at time t:

$$y_j^d(p_j(t), t) = g_j(t) + \int_0^1 c_{jk}(t)\, dk = \left[\frac{p_j(t)}{p(t)} \right]^{-\epsilon} y(t),$$

where $y(t) \equiv g(t) + \int_0^1 c_j(t)\, dj$ measures total consumption. The expression for $y_j^d(p_j(t), t)$ enters the Hamiltonian $\mathcal{H}_j$.

The terms featuring the bond holdings $b_j(t)$ in the Hamiltonian are the same as in online appendix A.1. Therefore, the optimality condition $\partial\mathcal{H}_j/\partial b_j = \delta\mathcal{A}_j - \dot{\mathcal{A}}_j$ remains the same, and the Euler equation (A7) remains valid. In equilibrium, the Euler equation simplifies to

$$\frac{\dot{c}}{c} = r - \delta + \sigma + u'(0)c. \tag{A34}$$

The terms featuring inflation $\pi_j(t)$ in the Hamiltonian are also the same as in online appendix A.1. Thus, the optimality condition $\partial\mathcal{H}_j/\partial\pi_j = 0$ is unchanged, and (A8) and (A9) hold.

Last, because the disutility from labor is convex, the optimality condition $\partial\mathcal{H}_j/\partial p_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j$ is modified. The condition now gives

$$\frac{\epsilon}{p_j}\left(\frac{\kappa}{a}y_j\right)^{1+\eta} + (1-\epsilon)\mathcal{A}_j y_j + \mathcal{B}_j\pi_j = \delta\mathcal{B}_j - \dot{\mathcal{B}}_j,$$

which we rewrite

$$\pi_j - \frac{(\epsilon-1)y_j\mathcal{A}_j}{\mathcal{B}_j p_j}\left[p_j - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta}\frac{y_j^\eta}{\mathcal{A}_j}\right] = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Combining this equation with (A4), (A8), and (A9), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon-1)y_j}{\gamma c_j\pi_j}\left[\frac{p_j}{p} - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta} y_j^\eta c_j\right]. \tag{A35}$$

In equilibrium, the Phillips curve simplifies to

$$\dot{\pi} = \delta\pi + \frac{(\epsilon-1)(c+g)}{\gamma c}\left[1 - \frac{\epsilon}{\epsilon-1}\left(\frac{\kappa}{a}\right)^{1+\eta}(c+g)^\eta c\right], \tag{A36}$$

where $c + g = y$ is aggregate output.

E.3. Linearized Euler-Phillips system

We now linearize the Euler-Phillips system around the natural steady state, which has zero inflation and no government spending. The analysis of the model with government spending is based on this linearized system.

Since $\dot{\pi} = \pi = g = 0$ at the natural steady state, (A36) implies that the natural level of consumption is

$$c^n = \left(\frac{\epsilon-1}{\epsilon}\right)^{1/(1+\eta)}\frac{a}{\kappa}.$$

Since $\dot{c} = 0$ and $c = c^n$ at the natural steady state, (A34) implies that the natural rate of interest is

$$r^n = \delta - \sigma - u'(0)c^n.$$

Euler equation. We first linearize the Euler equation (A34) around the point $[c = c^n, \pi = 0]$. We consider two different monetary-policy rules. First, when monetary policy is normal, $r(\pi) = r^n + (\phi - 1)\,\pi$. Then the Euler equation is $\dot{c} = E(c, \pi)$, where

$$E(c, \pi) = c\left[(\phi - 1)\pi + u'(0)(c - c^n)\right].$$

The linearized version is

$$\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi}\pi,$$

where the derivatives are evaluated at $[c = c^n, \pi = 0]$. We have

$$E(c^n, 0) = 0, \qquad \frac{\partial E}{\partial c} = c^n u'(0), \qquad \frac{\partial E}{\partial \pi} = c^n(\phi - 1).$$

So the linearized Euler equation with normal monetary policy is

$$\dot{c} = c^n\left[(\phi - 1)\pi + u'(0)(c - c^n)\right]. \tag{A37}$$

Second, when monetary policy is at the ZLB, $r(\pi) = -\pi$. Then the Euler equation becomes $\dot{c} = E(c, \pi)$ where

$$E(c, \pi) = c\left[-r^n - \pi + u'(0)(c - c^n)\right].$$

The linearized version is

$$\dot{c} = E(c^n, 0) + \frac{\partial E}{\partial c}(c - c^n) + \frac{\partial E}{\partial \pi}\pi,$$

where the derivatives are evaluated at $[c = c^n, \pi = 0]$. We have

$$E(c^n, 0) = -c^n r^n, \qquad \frac{\partial E}{\partial c} = c^n u'(0), \qquad \frac{\partial E}{\partial \pi} = -c^n.$$

So the linearized Euler equation at the ZLB is

$$\dot{c} = c^n\left[-r^n - \pi + u'(0)(c - c^n)\right]. \tag{A38}$$

In steady state, at the ZLB, the linearized Euler equation becomes

$$\pi = -r^n + u'(0)(c - c^n). \tag{A39}$$

Phillips curve. Next we linearize the Phillips curve (A36) around the point $[c = c^n, \pi = 0, g = 0]$. The Phillips curve can be written $\dot{\pi} = P(c, \pi, g)$ where

$$P(c, \pi, g) = \delta\pi + \frac{(\epsilon - 1)(c + g)}{\gamma c}\left[1 - \frac{\epsilon}{\epsilon - 1}\left(\frac{\kappa}{a}\right)^{1+\eta}(c + g)^{\eta} c\right].$$

The linearized version is

$$\dot{\pi} = P(c^n, 0, 0) + \frac{\partial P}{\partial c}(c - c^n) + \frac{\partial P}{\partial \pi}\pi + \frac{\partial P}{\partial g}g,$$

where the derivatives are evaluated at $[c = c^n, \pi = 0, g = 0]$. We have

$$\begin{aligned}
&P(c^n, 0, 0) = 0 \\
&\qquad \frac{\partial P}{\partial c} = -\frac{\epsilon}{\gamma}\left(\frac{\kappa}{a}\right)^{1+\eta}(1 + \eta)\,(c^n)^{\eta} = -(1 + \eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)} \\
&\qquad \frac{\partial P}{\partial \pi} = \delta \\
&\qquad \frac{\partial P}{\partial g} = -\frac{\epsilon}{\gamma}\left(\frac{\kappa}{a}\right)^{1+\eta}\eta\,(c^n)^{\eta} = -\eta\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}.
\end{aligned}$$

Hence, the linearized Phillips curve is

$$\dot{\pi} = \delta\pi - \frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\left[(1 + \eta)\,(c - c^n) + \eta g\right]. \tag{A40}$$

In steady state, the linearized Phillips curve becomes

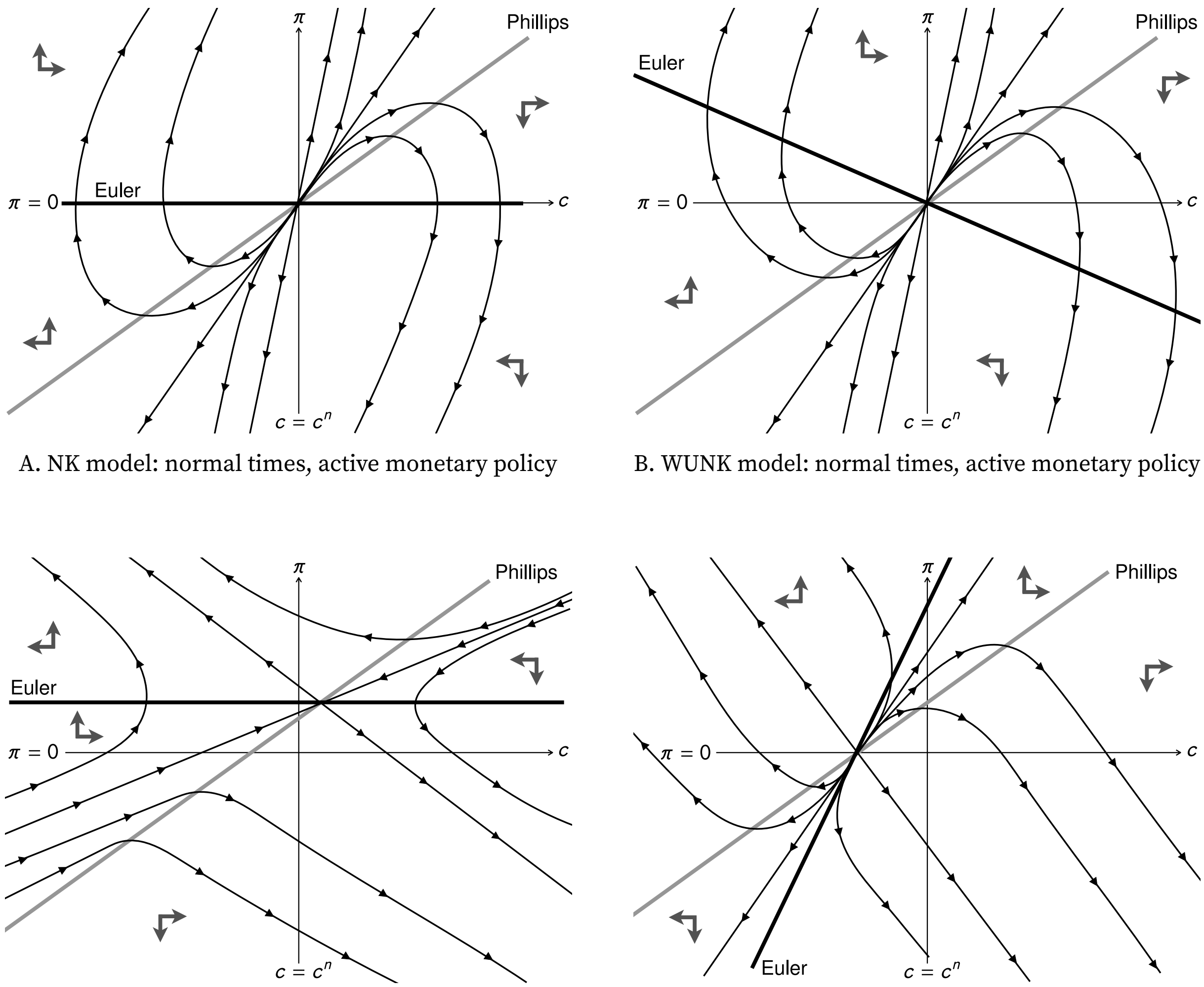
$$\pi = -\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\eta/(1+\eta)}\left[(1 + \eta)(c - c^n) + \eta g\right]. \tag{A41}$$

E.4. Phase diagrams

Using the linearized Euler-Phillips system, we construct the phase diagrams of the NK and WUNK models with government spending.

Normal times. We first construct the phase diagrams for normal times with active monetary policy. The linearized Euler-Phillips system is composed of (A37) with $\phi > 1$ and (A40) with $g = 0$.

We construct a phase diagram with private consumption c on the horizontal axis and inflation π on the vertical axis. We follow the methodology developed in section 3: we plot the loci $\dot{\pi} = 0$ and $\dot{c} = 0$, and then determine the sign of $\dot{\pi}$ and $\dot{c}$ in the four quadrants of the plan delimited by



A. NK model: normal times, active monetary policy

B. WUNK model: normal times, active monetary policy

C. NK model: ZLB

D. WUNK model: ZLB

FIGURE A1. Phase diagrams of the linearized Euler-Phillips system in the NK and WUNK models with government spending

The variable c is private consumption; π is inflation; c^n is the natural level of consumption. The Euler line is the locus $\dot{c} = 0$; the Phillips line is the locus $\dot{\pi} = 0$. The trajectories are solutions to the system, plotted for t going from $-\infty$ to $+\infty$. The NK model is the standard New Keynesian model. The WUNK model is the same model, except that the marginal utility of wealth is not zero but is sufficiently large to satisfy condition (12). In normal times with active monetary policy, the natural rate of interest r^n is positive, the monetary-policy rate is given by $i = r^n + \phi\pi$ with $\phi > 1$, and government spending is zero; the Euler-Phillips system is composed of (A37) with $\phi > 1$ and (A40) with $g = 0$. At the ZLB, the natural rate of interest is negative, the monetary-policy rate is zero, and government spending is positive; the Euler-Phillips system is composed of (A38) and (A40) with $g > 0$. The figure shows that in the NK model, the Euler-Phillips system is a source in normal times with active monetary policy (A); but the system is a saddle at the ZLB (C). In the WUNK model, by contrast, the Euler-Phillips system is a source both in normal times and at the ZLB (B, D).

the two loci. The resulting phase diagrams are displayed in the top panels of figure A1. They are similar to the phase diagrams in the basic model (figures 1A and 1B).[1]

The phase diagrams show that in normal times, with active monetary policy, the Euler-Phillips system is a source in the NK and WUNK models. An algebraic approach confirms this result. The linearized Euler-Phillips system is

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)c^n & (\phi-1)c^n \\ -(1+\eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)} & \delta \end{bmatrix} \begin{bmatrix} c - c^n \\ \pi \end{bmatrix}.$$

We denote the above matrix by $\boldsymbol{M}$. We classify the Euler-Phillips system using the trace and determinant of $\boldsymbol{M}$, as in online appendix D.1:

$$\begin{aligned} \operatorname{tr}(\boldsymbol{M}) &= \delta + u'(0)c^n \\ \det(\boldsymbol{M}) &= \delta c^n \left[u'(0) + (\phi-1)(1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\right]. \end{aligned}$$

In the NK model, $u'(0) = 0$ so $\operatorname{tr}(\boldsymbol{M}) = \delta > 0$ and the sign of $\det(\boldsymbol{M})$ is given by the sign of $\phi - 1$. Accordingly when monetary policy is active ($\phi > 1$), $\det(\boldsymbol{M}) > 0$: the Euler-Phillips system is a source. In contrast, when monetary policy is passive ($\phi < 1$), $\det(\boldsymbol{M}) < 0$: the Euler-Phillips system is a saddle.

In the WUNK model, $\operatorname{tr}(\boldsymbol{M}) > \delta > 0$. Moreover, $\phi - 1 \geq -1$ for any $\phi \geq 0$, so we have

$$\det(\boldsymbol{M}) \geq \delta c^n \left[u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\right].$$

The WUNK assumption (12) says that the term in square brackets is positive, so $\det(\boldsymbol{M}) > 0$. We conclude that the Euler-Phillips system is a source whether monetary policy is active or passive.

ZLB. We turn to the phase diagrams at the ZLB. The linearized Euler-Phillips system is composed of (A38) and (A40) with $g > 0$.

Once again, we follow the methodology developed in section 3 to construct the phase diagrams. The resulting phase diagrams are displayed in the bottom panels of figure A1. The diagrams have the same properties as in the basic model (figures 1C and 1D), but for one difference: the Phillips line shifts upward because government spending is positive. Hence, the Phillips line lies above the point $[c = c^n, \pi = 0]$. While this shift does not affect the classification of the Euler-Phillips

[1] The phase diagrams of figure 1 have output y on the horizontal axis instead of private consumption c. But $y = c$ in the basic model (government spending is zero), so phase diagrams with c on the horizontal axis would be identical.

system (source or saddle), it changes the location of the steady state. In fact, by solving the system given by (A39) and (A41), we find that private consumption and inflation at the ZLB steady state are

$$c^g = c^n + \frac{r^n + \frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\eta g}{u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}} \tag{A42}$$

$$\pi^g = \frac{(1+\eta)r^n + u'(0)\eta g}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)}. \tag{A43}$$

Steady-state consumption may be above or below natural consumption, depending on the amount of government spending. In the WUNK model, inflation may be positive or negative, depending on the amount of government spending.

The phase diagrams show that at the ZLB, the Euler-Phillips system is a source in the WUNK model but a saddle in the NK model. An algebraic approach confirms this classification. Rewritten in canonical form, the linearized Euler-Phillips system becomes

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)c^n & -c^n \\ -(1+\eta)\frac{\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)} & \delta \end{bmatrix} \begin{bmatrix} c - c^g \\ \pi - \pi^g \end{bmatrix}.$$

We denote the above matrix by $\boldsymbol{M}$. We classify the Euler-Phillips system using the trace and determinant of $\boldsymbol{M}$, as in online appendix D.1:

$$\begin{aligned} \operatorname{tr}(\boldsymbol{M}) &= \delta + u'(0)c^n \\ \det(\boldsymbol{M}) &= \delta c^n \left[u'(0) - (1+\eta)\frac{\epsilon\kappa}{\delta\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}\right]. \end{aligned}$$

In the NK model, $u'(0) = 0$ so $\det(\boldsymbol{M}) < 0$, indicating that the Euler-Phillips system is a saddle. In the WUNK model, condition (12) implies that $\det(\boldsymbol{M}) > 0$; since we also have $\operatorname{tr}(\boldsymbol{M}) > 0$, we conclude that the Euler-Phillips system is a source. We can also show that $\operatorname{tr}(\boldsymbol{M})^2 - 4\det(\boldsymbol{M}) > 0$, which indicates that the system is a nodal source, not a spiral source.

Appendix F. Proofs with government spending

We complement the proofs of propositions 5 and 9, which pertain to the model with government spending.

F.1. Complement to the proof of proposition 5

We characterize the amount g^* in the NK model, and we compute the limit of the government-spending multiplier in the WUNK model.

In the NK model, the amount g^* of government spending is the amount that makes the unstable line of the dynamical system go through the natural steady state. With less spending than g^* (figure 5B), the natural steady state is below the unstable line and is connected to trajectories coming from the southwest quadrant of the phase diagram. Hence, for $g < g^*$, $\lim_{T\to\infty} c(0; g) = -\infty$. With more spending than g^* (figure 5D), the natural steady state is above the unstable line and is connected to trajectories coming from the northeast quadrant. Hence, for $g > g^*$, $\lim_{T\to\infty} c(0; g) = +\infty$. Accordingly, for any $s > 0$, $\lim_{T\to\infty} m(g^*, s) = +\infty$.

In the WUNK model, when the ZLB is infinitely long-lasting, the economy jumps to the ZLB steady state at time 0: $\lim_{T\to\infty} c(0; g) = c^g(g)$, where $c^g(g)$ is given by (A42). The steady-state consumption $c^g(g)$ is linear in government spending g, with a coefficient in front of g of

$$\frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)}.$$

Accordingly, for any $s > 0$, we have

$$\begin{aligned}
\lim_{T\to\infty} m(g, s) &= 1 + \frac{\lim_{T\to\infty} c(0; g + s/2) - \lim_{T\to\infty} c(0; g - s/2)}{s} \\
&= 1 + \frac{c^g(g + s/2) - c^g(g - s/2)}{s} \\
&= 1 + \frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)} - (1+\eta)},
\end{aligned}$$

which corresponds to (13).

F.2. Complement to the proof of proposition 9

We compute the government-spending multiplier at the ZLB in the WUNK model. Private consumption and inflation at the ZLB steady state are determined by (A42) and (A43). The coefficients in front of government spending g in these expressions are

$$\frac{\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)}-(1+\eta)} \quad \text{and} \quad \frac{u'(0)\eta}{u'(0)\frac{\delta\gamma a}{\epsilon\kappa}\left(\frac{\epsilon}{\epsilon-1}\right)^{\eta/(1+\eta)}-(1+\eta)}.$$

Since (12) holds, both coefficients are positive. Hence, an increase in g raises private consumption and inflation. Moreover, dc/dg is given by the first of these coefficient, which immediately yields the expression for the multiplier $dy/dg = 1 + dc/dg$.

Appendix G. WUNK assumption in terms of estimable statistics

We re-express the WUNK assumption in terms of estimable statistics. We first work on the model with linear disutility of labor, in which the assumption is given by (9). We then turn to the model with convex disutility of labor, in which the assumption is given by (12).

G.1. Linear disutility of labor

When the disutility of labor is linear, the WUNK assumption is given by (9). Multiplying (9) by y^n, we obtain

$$u'(0)y^n > \frac{1}{\delta} \cdot \frac{y^n \epsilon\kappa}{\gamma a}.$$

The time discount rate δ has been estimated in numerous studies. We therefore only need to express $u'(0)y^n$ and $y^n\epsilon\kappa/(\gamma a)$ in terms of estimable statistics.

First, the definition of the natural rate of interest, given by (5), implies that $u'(0)y^n = \delta - \sigma - r^n$. Following the New Keynesian literature, we set the financial-intermediation spread to $\sigma = 0$ in normal times (Woodford 2011, p. 20). Hence, in normal times, $u'(0)y^n = \delta - r^n$. Thus, $u'(0)y^n$ can be measured from the gap between the discount rate δ and the average natural rate of interest r^n—both of which have been estimated by many studies.

Second, we show that $y^n\epsilon\kappa/(\gamma a)$ can be measured from estimates of the New Keynesian Phillips curve. To establish this, we compute the discrete-time New Keynesian Phillips curve arising from our continuous-time model. We start from the first-order approximation

$$\pi(t) = \pi(t + dt) - \dot{\pi}(t + dt)dt$$

and use (1) to measure $\dot{\pi}(t + dt)$. We obtain

$$\pi(t) = \pi(t + dt) - \delta\pi(t + dt)dt + \frac{y^n \epsilon\kappa}{\gamma a} \cdot \frac{y(t) - y^n}{y^n} dt.$$

(We have replaced $y(t + dt)dt$ by $y(t)dt$ since the difference between the two is of second order.) Setting the unit of time to one quarter (as in the empirical literature) and $dt = 1$, we obtain

$$\pi(t) = (1 - \delta)\pi(t + 1) + \frac{y^n \epsilon\kappa}{\gamma a} x(t), \tag{A44}$$

where $\pi(t)$ is quarterly inflation at time t, $\pi(t + 1)$ is quarterly inflation at time $t + 1$, and

$$x(t) = \frac{y(t) - y^n}{y^n}$$

is the output gap at time t. Equation (A44) is a typical New Keynesian Phillips curve, so we can measure $y^n\epsilon\kappa/(\gamma a)$ by estimating the coefficient on output gap in a standard New Keynesian Phillips curve—which has been done many times.

To sum up, we rewrite the WUNK assumption as

$$\delta - r^n > \frac{\lambda}{\delta},$$

where δ is the time discount rate, r^n is the average natural interest rate, and λ is the output-gap coefficient in a standard New Keynesian Phillips curve. This is just (14).

G.2. Convex disutility of labor

When the disutility of labor is convex, the WUNK assumption is given by (12):

$$u'(0)y^n > \frac{1}{\delta} \cdot \frac{y^n\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}(1+\eta).$$

To rewrite this condition in terms of estimating statistics, we follow the previous method. The only change occurs when computing the discrete-time New Keynesian Phillips curve arising from the continuous-time model. To measure $\dot{\pi}(t+dt)$, we use (A40) with $g=0$ and $c=y$. As a result, (A44) becomes

$$\pi(t) = (1-\delta)\pi(t+1) + \frac{y^n\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}(1+\eta)x(t),$$

where $\pi(t)$ and $\pi(t+1)$ are quarterly inflation rates and $x(t)$ is the output gap. This is just a typical New Keynesian Phillips curve. Hence, again, we can measure

$$\frac{y^n\epsilon\kappa}{\gamma a}\left(\frac{\epsilon-1}{\epsilon}\right)^{\eta/(1+\eta)}(1+\eta)$$

by estimating the output-gap coefficient in a standard New Keynesian Phillips curve.

To conclude, just as with a linear disutility of labor, we can write the WUNK assumption as

$$\delta - r^n > \frac{\lambda}{\delta},$$

where δ is the time discount rate, r^n is the average natural rate of interest, and λ is the output-gap coefficient in a standard New Keynesian Phillips curve.

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